Inference Methods for Latent Dirichlet Allocation

Chase Geigle
University of Illinois at Urbana-Champaign
Department of Computer Science
geigle1@illinois.edu

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Abstract

Latent Dirichlet Allocation (LDA) has seen a huge number of works surrounding it in recent years in the machine learning and text mining communities. Numerous inference algorithms for the model have been introduced, each with its trade-offs. In this survey, we investigate some of the main strategies that have been applied to inference in this model and summarize the current state-of-the-art in LDA inference methods.

1 The Dirichlet Distribution and its Relation to the Multinomial

Before exploring Latent **Dirichlet** Allocation in depth, it is important to understand some properties of the Dirichlet distribution it uses as a component.

The Dirichlet distribution with parameter vector α of length K is defined as

$$Dirichlet(\theta;\alpha) = \frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_i^{\alpha_i - 1} \tag{1}$$

where $B(\alpha)$ is the multivariate Beta function, which can be expressed using the gamma function as

$$B(\alpha) = \frac{\prod_{i=1}^{K}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{K}\alpha_i\right)}. \tag{2}$$

The Dirichlet provides a distribution over vectors $\mathbf{x}$ that lie on the $(k-1)$-simplex. This is a complicated way of saying that the Dirichlet is a distribution over vectors $\theta \in \mathbb{R}^k$ such that the values $\theta_i \in [0,1]$ and $||\theta||_1 = 1$ (the values in θ sum to 1). In other words, **the Dirichlet is a distribution over the possible parameter vectors for a Multinomial distribution**. This fact is used in Latent Dirichlet allocation (hence its name) to provide a principled way of generating the multinomial distributions that comprise the word distributions for the topics as well as the topic proportions within each document.

The Dirichlet distribution, in addition to generating proper parameter vectors for a Multinomial, can be shown to be what is called a **conjugate prior** to the Multinomial. This means that if one were to use a Dirichlet distribution as a prior over the parameters of a Multinomial distribution, the resulting posterior distribution is *also* a Dirichlet distribution. We can see this as follows: let X be some data, θ be the parameters for a multinomial distribution, and $\theta \sim Dirichlet(\alpha)$ (that is, the prior over θ is a Dirichlet with parameter vector α). Let n_i be the number of times we observe value i in the data X. We can then see that

$$P(\theta \mid X, \alpha) \propto P(X \mid \theta) P(\theta \mid \alpha) \tag{3}$$

by Bayes' rule, and thus

$$P(\theta \mid X, \alpha) \propto \left(\prod_{i=1}^{N} p(x_i \mid \theta) \right) \left(\frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_i^{\alpha_i - 1} \right). \tag{4}$$

We can rewrite this as

$$P(\theta \mid X, \alpha) \propto \left(\prod_{i=1}^{K} \theta_i^{n_i} \right) \left(\frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_i^{\alpha_i - 1} \right) \tag{5}$$

and thus

$$P(\theta \mid X, \alpha) \propto \frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_i^{n_i + \alpha_i - 1} \tag{6}$$

which has the form of a Dirichlet distribution. Specifically, we know that this probability must be

$$P(\theta \mid X, \alpha) = \frac{1}{B(\alpha + \mathbf{n})} \prod_{i=1}^{K} \theta_i^{n_i + \alpha_i - 1}, \tag{7}$$

where $\mathbf{n}$ represents the vector of count data we obtained from X, in order to integrate to unity (and thus be a properly normalized probability distribution). Thus, the posterior distribution $P(\theta \mid X, \alpha)$ is itself $Dirichlet(\alpha + \mathbf{n})$.

Since we know that this is a distribution, we have that

$$\int \frac{1}{B(\alpha + \mathbf{n})} \prod_{i=1}^{K} \theta_i^{n_i + \alpha_i - 1} d\theta = 1$$

and thus

$$\frac{1}{B(\alpha + \mathbf{n})} \int \prod_{i=1}^{K} \theta_i^{n_i + \alpha_i - 1} d\theta = 1$$

This implies

$$\int \prod_{i=1}^{K} \theta_i^{n_i+\alpha_i-1} d\theta = B(\alpha + \mathbf{n}) \tag{8}$$

which is a useful property we will use later.

Finally, we note that the Dirichlet distribution is a member of what is called the **exponential family**. Distributions in this family can be written in the following common form

$$P(\theta \mid \eta) = h(\theta)\exp\{\eta^T t(\theta) - a(\eta)\}, \tag{9}$$

where η is called the natural parameter, $t(\theta)$ is the sufficient statistic, $h(\theta)$ is the underlying measure, and $a(\eta)$ is the log normalizer

$$a(\eta) = \log \int h(\theta)\exp\{\eta^T t(\theta)\} d\theta. \tag{10}$$

We can show that the Dirichlet is in fact a member of the exponential family by exponentiating the log of the PDF defined in Equation 1:

$$P(\theta \mid \alpha) = \exp\left\{\left(\sum_{i=1}^{K}(\alpha_i - 1)\log\theta_i\right) - \log B(\alpha)\right\} \tag{11}$$

where we can now note that the natural parameter is $\eta_i = \alpha_i - 1$, the sufficient statistic is $t(\theta_i) = \log\theta_i$, and the log normalizer is $a(\eta) = \log B(\alpha)$.

It turns out that the derivatives of the log normalizer $a(\eta)$ are the moments of the sufficient statistic. Thus,

$$E_P[t(\theta)] = \frac{\partial a}{\partial \eta^T}, \tag{12}$$

which is a useful fact we will use later.

2 Latent Dirichlet Allocation

The model for Latent Dirichlet Allocation was first introduced Blei, Ng, and Jordan [2], and is a generative model which models documents as mixtures of topics. Formally, the generative model looks like this, assuming one has K topics, a corpus D of $M = |D|$ documents, and a vocabulary consisting of V unique words:

- For $j \in [1, \ldots, M]$,
  - $\theta_j \sim Dirichlet(\alpha)$
  - For $t \in [1, \ldots, |d_j|]$
    - $z_{j,t} \sim Multinomial(\theta_j)$
    - $w_{j,t} \sim Multinomial(\phi_{z_{j,t}})$

In words, this means that there are K topics $\phi_{1,\ldots,K}$ that are *shared* among all documents, and each document d_j in the corpus D is considered as a mixture over these topics, indicated by θ_j. Then, we can generate the words for document d_j by first sampling a topic assignment $z_{j,t}$ from the topic proportions θ_j, and then sampling a word from the corresponding topic $\phi_{z_{j,t}}$. $z_{j,t}$ is then an indicator variable that denotes which topic from $1, \ldots K$ was selected for the t-th word in d_j. The graphic model representation for this generative process is given in Figure 1.

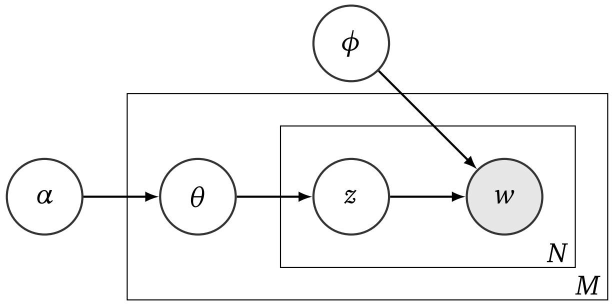

Figure 1: Graphical model for LDA, as described in [2].

It is important to point out some key assumptions with this model. First, we assume that the number of topics K is a fixed quantity known in advance, and that each ϕ_k is a fixed quantity to be estimated. Furthermore, we assume that the number of unique words V fixed and known in advance (that is, the model lacks any mechanisms for generating "new words"). Each word within a document is independent (encoding the traditional "bag of words" assumption), and each topic proportion θ_j is independent.

In this formulation, we can see that the joint distribution of the topic mixtures Θ, the set of topic assignments $\mathbf{Z}$, and the words in the corpus $\mathbf{W}$ given the hyperparameter α and the topics Φ is given by

$$P(\mathbf{W}, \mathbf{Z}, \Theta \mid \alpha, \Phi) = \prod_{j=1}^{M} P(\theta_j \mid \alpha) \prod_{t=1}^{N_j} P(z_{j,t} \mid \theta_j) P(w_{j,t} \mid \phi_{z_{j,t}}). \tag{13}$$

Most works now do not actually use this original formulation, as it has a weakness in that it does not also place a prior on the each ϕ_k—since this quantity is not modeled in the machinery for inference, it must be estimated using maximum likelihood. Choosing another Dirichlet parameterized by β as the prior for each ϕ_k, the generative model becomes:

1. For $i \in [1, \ldots K]$, $\phi_i \sim Dirichlet(\beta)$
2. For $j \in [1, \ldots, M]$,
   - $\theta_j \sim Dirichlet(\alpha)$
   - For $t \in [1, \ldots, |d_j|]$
     - $z_{j,t} \sim Multinomial(\theta_j)$
     - $w_{j,t} \sim Multinomial(\phi_{z_{j,t}})$

The graphic model representation for this is given in Figure 2. Blei et al. refer to this model as "smoothed LDA" in their work.

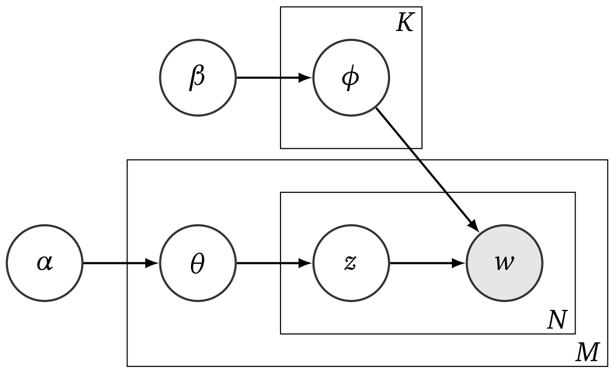

Figure 2: Graphical model for "smoothed" LDA, as described in Blei et al. [2].

In this "smoothed" formulation (which lends itself to a more fully Bayesian inference approach), we can model the joint distribution of the topic mixtures Θ, the set of topic assignments $\mathbf{Z}$, the words of the corpus $\mathbf{W}$, and the topics Φ by

$$P(\mathbf{W},\mathbf{Z},\Theta,\Phi \mid \alpha,\beta) = \prod_{i=1}^{K} P(\phi_k \mid \beta) \times \prod_{j=1}^{M} P(\theta_j \mid \alpha) \prod_{t=1}^{N_j} P(z_{j,t} \mid \theta_j) P(w_{j,t} \mid \phi_{z_{j,t}}). \tag{14}$$

As mentioned, most approaches take the approach of Equation 14, but for the sake of completeness we will also give the original formulation for inference for the model given by Equation 13.

3 LDA vs PLSA

Let's compare the LDA model with the PLSA model introduced by Hofmann [7], as there are critical differences that motivate all of the approximate inference algorithms for LDA.

In PLSA, we assume that the topic word distributions ϕ_i and the document topic proportions θ_j[1] are *parameters* in the model. By comparison, (the smoothed version of) LDA treats each ϕ_i and θ_j as *latent variables*. This small difference has a dramatic impact on the way we infer the quantities of interest in topic models. After all, the quantities we are interested in are the distributions Φ and Θ. We can use the EM algorithm to find the maximum likelihood estimate for these quantities in PLSA, since they are modeled as parameters. After our EM algorithm converges, we can simply inspect the learned parameters to accomplish our goal of finding the topics and their coverage across documents in a corpus.

In LDA, however, these quantities must be *inferred* using Bayesian inference because they themselves are *latent variables*, just like the $\mathbf{Z}$ are inferred in PLSA. Specifically, in LDA we are interested in $P(\mathbf{Z},\Theta,\Phi \mid \mathbf{W},\alpha,\beta)$, the posterior distribution of the latent variables given the parameters α and β and our observed

[1]These are often called θ_i and π_j, respectively, in the standard PLSA notation. We will use the LDA notation throughout this note.

data $\mathbf{W}$. If we write this as

$$P(\mathbf{Z}, \Theta, \Phi \mid \mathbf{W}, \alpha, \beta) = \frac{P(\mathbf{W}, \mathbf{Z}, \Theta, \Phi \mid \alpha, \beta)}{P(\mathbf{W} \mid \alpha, \beta)}. \tag{15}$$

we can see that this distribution is intractable by looking at the form of the denominator

$$\begin{aligned} P(\mathbf{W} \mid \alpha, \beta) &= \int_\Phi \int_\Theta \sum_{\mathbf{Z}} P(\mathbf{W}, \mathbf{Z}, \Theta, \Phi \mid \alpha, \beta) d\Theta d\Phi \\ &= \int_\Phi p(\Phi \mid \beta) \int_\Theta p(\Theta \mid \alpha) \sum_{\mathbf{Z}} p(\mathbf{Z} \mid \Theta) p(\mathbf{W} \mid \mathbf{Z}, \Phi) d\Theta d\Phi \end{aligned}$$

and observing the coupling between Θ and Φ in the summation over the latent topic assignments. Thus, we are forced to turn to approximate inference methods to compute the posterior distribution over the latent variables we care about[2].

Why go through all this trouble? One of the main flaws of PLSA as pointed out by Blei et al. [2] is that PLSA is not a *fully generative* model in the sense that you cannot use PLSA to create new documents, as the topic proportion parameters are specific to each document in the corpus. To generate a new document, we require some way to arrive at this parameter vector for the new document, which PLSA does not provide. Thus, to adapt to new documents, PLSA has to use a heuristic where the new document is "folded in" and EM is re-run (holding the old parameters fixed) to estimate the topic proportion parameter for this new document. This is not probabilistically well motivated. LDA, on the other hand, provides a complete generative model for the brand new document by assuming that the topic proportions for this document are drawn from a Dirichlet distribution.

In practice, Lu et al. [8] showed PLSA and LDA tend to perform similarly when used as a component in a downstream task (like clustering or retrieval), with LDA having a slight advantage for document classification due to its "smoothed" nature causing it to avoid overfitting. However, if your goal is to simply discover the topics present and their coverage in a *fixed corpus*, the difference between PLSA and LDA is often negligible.

4 Variational Inference for LDA

4.1 Original (Un-smoothed) Formulation

Blei et al. [2] give an inference method based on *variational inference* to approximate the posterior distribution of interest. The key idea here is to design a family of distributions Q that are tractable and have parameters which can be tuned to approximate the desired posterior P. The approach taken in the paper is often referred to as a *mean field approximation*, where they consider a family of distributions

[2] If we are *also* interested in maximum likelihood estimates for the two parameter vectors α and β, we can use the EM algorithm where the inference method slots in as the computation to be performed during the E-step. This results in an *empirical Bayes* algorithm often called *variational EM*.

Q that are fully factorized. In particular, for LDA they derive the variational distribution as

$$Q(\mathbf{Z}, \Theta \mid \gamma, \pi) = \prod_{j=1}^{M} q_j(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \tag{16}$$

$$= \prod_{j=1}^{M} q_j(\theta_j \mid \gamma_j) \prod_{t=1}^{N_j} q_j(z_{j,t} \mid \pi_{j,t}) \tag{17}$$

where γ_j and π_j are free variational parameters for the variational distribution $q_j(\bullet)$ for document j. The graphic model representation for this factorized variational distribution is given in Figure 3.

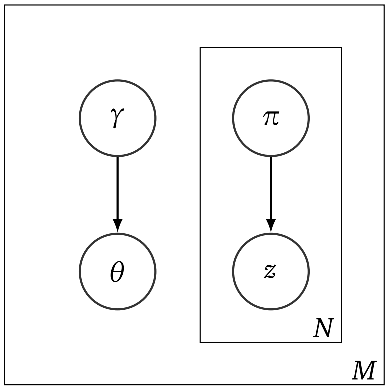

Figure 3: Graphical model for the factorized variational distribution in Blei et al. [2].

Inference is then performed by minimizing the Kullback-Leibler (KL) divergence between the variational distributions $q_j(\bullet)$ and the true posteriors $p(\theta_j, \mathbf{z}_j \mid \mathbf{w}_j, \alpha, \Phi)$. If you are interested only in the algorithm, and not its derivation, you may skip ahead to section 4.1.2.

4.1.1 Variational Inference Algorithm Derivation

Since the distribution is fully factorized across documents in both cases, optimizing the parameters for each document in turn will optimize the distribution as a whole, so we will focus on the document level here.

Focusing on document j, the KL divergence of p from q is

$$D(q \,||\, p) = \int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \log \frac{q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j)}{p(\mathbf{z}_j, \theta_j \mid \mathbf{w}_j, \alpha, \Phi)} d\theta_j. \tag{18}$$

This can be rewritten as

$$
\begin{aligned}
&= \int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \log \frac{q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) p(\mathbf{w}_j \mid \alpha, \Phi)}{p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi)} d\theta_j \\
&= \int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \log \frac{q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j)}{p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi)} d\theta_j + \log p(\mathbf{w}_j \mid \alpha, \Phi).
\end{aligned}
$$

Noting that $\log p(\mathbf{w}_j \mid \alpha, \Phi)$ is fixed with respect to q, we then want to minimize the first term in the expression

$$
\begin{aligned}
&\int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \log \frac{q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j)}{p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi)} d\theta_j \\
&\quad = \int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \Theta \mid \gamma_j, \phi_j) \log q(\mathbf{z}_j, \Theta \mid \gamma_j, \pi_j) d\theta_j - \int_{\theta_j} \sum_{\mathbf{z}_j} q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \log p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi) d\theta_j \\
&\quad = E_q \left[\log q(\mathbf{z}_j, \theta_j \mid \gamma_j, \pi_j) \right] - E_q \left[\log p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi) \right].
\end{aligned}
$$

Thus, if we define

$$
\begin{aligned}
L(\gamma_j, \pi_j \mid \alpha, \Phi) &= E_q \left[\log p(\mathbf{w}_j, \mathbf{z}_j, \theta_j \mid \alpha, \Phi) \right] - E_q \left[\log q(\mathbf{z}_j, \Theta \mid \gamma_j, \pi_j) \right] \\
&= E_q \left[\log p(\theta_j \mid \alpha) \right] + E_q \left[\log p(\mathbf{z}_j \mid \theta_j) \right] + E_q \left[\log p(\mathbf{w}_j \mid \mathbf{z}_j, \Phi) \right] \qquad (19) \\
&\quad - E_q \left[\log q(\theta_j \mid \gamma_j) \right] - E_q \left[\log q(\mathbf{z}_j \mid \pi_j) \right]
\end{aligned}
$$

then we can see that maximizing $L(\bullet)$ will minimize $D(q \mid\mid p)$. Proceeding to simplify each of the expectations in $L(\bullet)$, we first have

$$
\begin{aligned}
E_q \left[\log p(\theta_j \mid \alpha) \right] &= E_q \left[\log \left(\frac{\Gamma(\sum_{i=1}^K \alpha_i)}{\prod_{i=1}^K \Gamma(\alpha_i)} \prod_{i=1}^K \theta_{j,i}^{\alpha_i - 1} \right) \right] \qquad (20) \\
&= E_q \left[\log \Gamma \left(\sum_{i=1}^K \alpha_i \right) - \sum_{i=1}^k \log \Gamma(\alpha_i) + \sum_{i=1}^k (\alpha_i - 1) \log \theta_{j,i} \right] \qquad (21) \\
&= \log \Gamma \left(\sum_{i=1}^K \alpha_i \right) - \sum_{i=1}^k \log \Gamma(\alpha_i) + \sum_{i=1}^k (\alpha_i - 1) E_q \left[\log \theta_{j,i} \right]. \qquad (22)
\end{aligned}
$$

We're left with a single expectation $E_q[\log \theta_{j,i}]$. We can now use the fact that $\theta_j \sim Dirichlet(\gamma_j)$, which is a member of the exponential family (see Equation 11), to solve for this expectation. Recall that the sufficient statistic for the above Dirichlet is $\log \theta_{j,i}$ and that it has a log normalizer of

$$
a(\gamma_j) = \sum_{i=1}^k \log \Gamma(\gamma_{j,i}) - \log \Gamma \left(\sum_{i=1} \gamma_{j,i} \right). \qquad (23)
$$

Thus,

$$E_q\left[\log\theta_{j,i}\right] = \frac{d}{d\gamma_{j,i}} a(\gamma_j) \tag{24}$$

$$= \frac{d}{d\gamma_{j,i}} \sum_{k=1}^{K} \log\Gamma(\gamma_{j,k}) - \log\Gamma\left(\sum_{l=1}^{K}\gamma_{j,l}\right) \tag{25}$$

$$= \Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right) \tag{26}$$

where $\Psi(\bullet)$ is the "digamma" function, the first derivative of the log Gamma function. Plugging this in, we arrive at

$$E_q\left[\log p(\theta_j \mid \alpha)\right] = \log\Gamma\left(\sum_{i=1}^{K}\alpha_i\right) - \sum_{i=1}^{k}\log\Gamma(\alpha_i) + \sum_{i=1}^{k}(\alpha_i - 1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right) \tag{27}$$

Next, we turn our attention to the second term of $L(\bullet)$. Let $\mathbb{1}(\bullet)$ be an indicator function equal to 1 if the condition is true. Then, we have

$$E_q\left[\log p(z_j \mid \theta_j)\right] = E_q\left[\sum_{t=1}^{N_j}\sum_{i=1}^{K}\mathbb{1}(z_{j,t} = i)\log\theta_{j,i}\right] \tag{28}$$

$$= \sum_{t=1}^{N_j}\sum_{i=1}^{K} E_q[\mathbb{1}(z_{j,t} = i)]E_q[\log\theta_{j,i}] \tag{29}$$

$$= \sum_{t=1}^{N_j}\sum_{i=1}^{K}\pi_{j,t,i}\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right), \tag{30}$$

where we have again used equations 24–26 for simplifying $E_q[\log\theta_{j,i}]$. The third term can be simplified as follows:

$$E_q\left[\log p(w_i \mid z_j, \Phi)\right] = E_q\left[\sum_{t=1}^{N_j}\sum_{i=1}^{K}\sum_{r=1}^{V}\mathbb{1}(z_{j,t} = i)\mathbb{1}(w_{j,t} = r)\log\phi_{i,r}\right] \tag{31}$$

$$= \sum_{t=1}^{N_j}\sum_{i=1}^{K}\sum_{r=1}^{V} E_q[\mathbb{1}(z_{j,t} = i)\mathbb{1}(w_{j,t} = r)\log\phi_{i,r}] \tag{32}$$

$$= \sum_{t=1}^{N_j}\sum_{i=1}^{K}\sum_{r=1}^{V}\pi_{j,t,i}\mathbb{1}(w_{j,t} = r)\log\phi_{i,r}. \tag{33}$$

The fourth term is very similar to the first, and is

$$E_q\left[\log q(\theta_j \mid \gamma_j)\right] = \log\Gamma\left(\sum_{i=1}^{K}\gamma_{j,i}\right) - \sum_{i=1}^{K}\log\Gamma(\gamma_{j,i}) + \sum_{i=1}^{K}(\gamma_{j,i} - 1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right) \tag{34}$$

The fifth and final term can be reduced as

$$
\begin{aligned}
E_q\left[\log q(z_j \mid \pi_j)\right] &= E_q\left[\sum_{t=1}^{N_j}\sum_{i=1}^{K} \mathbb{1}(z_{j,t}=i)\log \pi_{j,t,i}\right] && (35)\\
&= \sum_{t=1}^{N_j}\sum_{i=1}^{K} E_q[\mathbb{1}(z_{j,t}=i)]\log \pi_{j,t,i} && (36)\\
&= \sum_{t=1}^{N_j}\sum_{i=1}^{K} \pi_{j,t,i}\log \pi_{j,t,i}, && (37)
\end{aligned}
$$

and thus we arrive at

$$
\begin{aligned}
L(\gamma_j,\pi_j \mid \alpha,\Phi) = {} & \log\Gamma\left(\sum_{i=1}^{K}\alpha_i\right) - \sum_{i=1}^{k}\log\Gamma(\alpha_i) + \sum_{i=1}^{k}(\alpha_i - 1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right)\\
& + \sum_{t=1}^{N_j}\sum_{i=1}^{K}\pi_{j,t,i}\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right)\\
& + \sum_{t=1}^{N_j}\sum_{i=1}^{K}\sum_{r=1}^{V}\pi_{j,t,i}\mathbb{1}(w_{j,t}=r)\log\phi_{i,r} && (38)\\
& - \log\Gamma\left(\sum_{i=1}^{K}\gamma_{j,i}\right) + \sum_{i=1}^{K}\log\Gamma(\gamma_{j,i}) - \sum_{i=1}^{K}(\gamma_{j,i}-1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right)\\
& - \sum_{t=1}^{N_j}\sum_{i=1}^{K}\pi_{j,t,i}\log\pi_{j,t,i}.
\end{aligned}
$$

We can now perform a constrained maximization of $L(\bullet)$ by isolating terms and introducing Lagrange multipliers. We start by isolating the terms containing a single $\pi_{j,t,i}$. Noting the constraint here is $\sum_{l=1}^{K}\pi_{j,t,l} = 1$, we have

$$
L_{[\pi_{j,t,i}]} = \pi_{j,t,i}\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right) + \pi_{j,t,i}\log\phi_{i,w_{j,t}} - \pi_{j,t,i}\log\pi_{j,t,i} + \lambda\left(\sum_{l=1}^{K}\pi_{j,t,l} - 1\right) \quad (39)
$$

Taking derivatives with respect to $\pi_{j,t,l}$, we have

$$
\frac{\partial L}{\partial \pi_{j,t,i}} = \Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right) + \log\phi_{i,w_{j,t}} - \log\pi_{j,t,i} - 1 + \lambda. \quad (40)
$$

Setting this equation to zero and solving for $\pi_{j,t,i}$, we arrive at the update

$$
\pi_{j,t,i} \propto \phi_{i,w_{j,t}} \exp\left\{\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,i}\right)\right\} \quad (41)
$$

Concerning ourselves now with $\gamma_{j,i}$, which is unconstrained, we have

$$\begin{aligned}L_{[\gamma_j]} = &\sum_{i=1}^{k}(\alpha_i - 1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right)\\ &+ \sum_{t=1}^{N_j}\sum_{i=1}^{K}\pi_{j,t,i}\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right)\\ &- \log\Gamma\left(\sum_{i=1}^{K}\gamma_{j,i}\right) + \sum_{i=1}^{K}\log\Gamma(\gamma_{j,i}) - \sum_{i=1}^{K}(\gamma_{j,i} - 1)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right),\end{aligned} \tag{42}$$

which can be simplified to

$$L_{[\gamma_j]} = \sum_{i=1}^{K}\left(\alpha_i + \sum_{t=1}^{N_j}\pi_{j,t,i} - \gamma_{j,i}\right)\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\right) - \log\Gamma\left(\sum_{l=1}^{K}\gamma_{j,l}\right) + \sum_{i=1}^{K}\log\Gamma(\gamma_{j,i}) \tag{43}$$

Deriving with respect to $\gamma_{j,i}$, we get

$$\begin{aligned}\frac{\partial L}{\partial\gamma_{j,i}} = &\frac{\partial}{\partial\gamma_{j,i}}\sum_{l=1}^{K}\left(\alpha_l + \sum_{t=1}^{N_j}\pi_{j,t,l} - \gamma_{j,l}\right)\Psi(\gamma_{j,l}) - \sum_{l=1}^{K}\left(\alpha_l + \sum_{t=1}^{N_j}\pi_{j,t,l} - \gamma_{j,l}\right)\Psi(\sum_{l=1}^{K}\gamma_{j,l})\\ &- \log\Gamma\left(\sum_{l=1}^{K}\gamma_{j,l}\right) + \sum_{l=1}^{K}\log\Gamma\left(\gamma_{j,l}\right)\\ = &-\Psi(\gamma_{j,i}) + \Psi'(\gamma_{j,i})\left(\alpha_i + \sum_{t=1}^{N_j}\pi_{j,t,i} - \gamma_{j,i}\right)\\ &+ \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right) - \Psi'\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\sum_{l=1}^{K}\left(\alpha_l + \sum_{t=1}^{N_j}\pi_{j,t,i} - \gamma_{j,l}\right)\\ &- \Psi\left(\sum_{l=1}^{K}\gamma_{j,l}\right) + \Psi(\gamma_{j,i})\\ = &\Psi'(\gamma_{j,i})\left(\alpha_i + \sum_{t=1}^{N_j}\pi_{j,t,i} - \gamma_{j,i}\right) - \Psi'\left(\sum_{l=1}^{K}\gamma_{j,l}\right)\sum_{l=1}^{K}\left(\alpha_l + \sum_{t=1}^{N_j}\pi_{j,t,i} - \gamma_{j,l}\right).\end{aligned}$$

Setting this equal to zero and taking the obvious solution, we arrive at

$$\gamma_{j,i} = \alpha_i + \sum_{t=1}^{N_j}\pi_{j,t,i}. \tag{44}$$

4.1.2 Variational Inference Algorithm

The following updates have been derived to minimize the KL divergence from p to q for each document j:

$$\pi_{j,t,i} \propto \phi_{i,w_{j,t}} \exp\left(\Psi(\gamma_{j,i}) - \Psi\left(\sum_{k=1}^{K} \gamma_{j,k}\right)\right) \tag{45}$$

$$\gamma_{j,i} = \alpha_i + \sum_{t=1}^{N_j} \pi_{j,t,i}. \tag{46}$$

where $\Psi(\bullet)$ is the "digamma" function. It is important to note here that this inference process is done on a per-document level (hence the subscript j in all of these parameters). The authors speculate that the number of iterations required is linear in the length of the document (N_j), and the number of operations within the method is $O(Nk)$—thus they arrive at the running time estimate $O(N_j^2 k)$. However, since this procedure would need to be run for every document j, the full running time is multiplied by another factor of M.

While the procedure's found parameters γ_j can be used in place of the model's latent variables θ_j, it has no provision for directly finding the model parameters Φ, the actual topics. The authors propose a *variational expectation maximization* (variational EM) for estimating this parameter. The essential idea is this: run the previously described variational inference method for each document j until convergence. Then, using these fixed variational parameters, update the estimates for the model parameter Φ given these fixed variational parameters. These two steps are done iteratively until the lower bound on the log likelihood converges.

Specifically, the variational EM algorithm derived is the following:

E-step: Minimize $KL(q \,||\, p)$ by performing the following updates until convergence:

$$\pi_{j,t,i} \propto \phi_{i,w_{j,t}} \exp\left(\psi(\gamma_{j,i}) - \psi\left(\sum_{k=1}^{K} \gamma_{j,k}\right)\right)$$

$$\gamma_{j,i} = \alpha_i + \sum_{t=1}^{N_j} \pi_{j,t,i}.$$

q now is a good approximation to the posterior distribution p.

M-step: Using q, re-estimate the parameters Φ. Specifically, since $\pi_{j,t,i}$ represents the probability that word $w_{j,t}$ was assigned to topic i, we can compute and re-normalize expected counts:

$$\phi_{i,v} \propto \sum_{j=1}^{M} \sum_{t=1}^{N_j} \pi_{j,t,i} \mathbb{1}(w_{j,t} = v) \tag{47}$$

where $\mathbb{1}(\bullet)$ is the indicator function that takes value 1 if the condition is true, and value 0 otherwise.

4.2 Smoothed Formulation

In their smoothed formulation (see Figure 2), Blei et al. [2] find a factorized distribution which also contains Φ as a random variable with a Dirichlet prior with hyperparameter β as

$$Q(\Phi, \mathbf{Z}, \Theta \mid \lambda, \pi, \gamma) = \prod_{i=1}^{K} Dir(\phi_i \mid \lambda_i) \prod_{j=1}^{M} q_j(\theta_j, \mathbf{z}_j \mid \pi_j, \gamma_j) \tag{48}$$

The graphic model for this variational distribution is given in Figure 4.

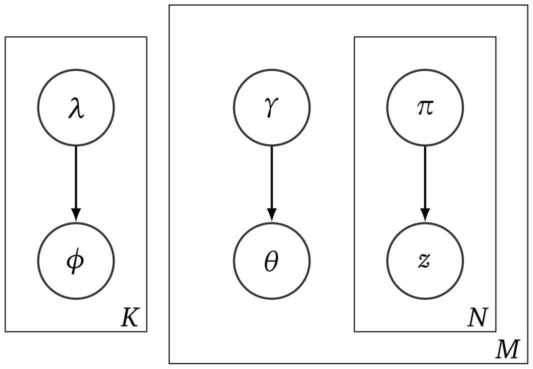

Figure 4: Graphical model for the factorized variational distribution for "smoothed LDA" [2].

The updating equations for γ and π remain the same as in the previous formulation, but they add an additional update for the new variational parameters λ as

$$\lambda_{i,v} = \beta_v + \sum_{j=1}^{M} \sum_{t=1}^{N_j} \pi_{j,t,v} \mathbb{1}(w_{j,t} = v). \tag{49}$$

This bears a striking resemblance to the update for $\phi_{i,v}$ in the variational EM algorithm given in the previous section's Equation 47, but with the addition of the prior's pseudo-counts β_v.

It is worth noting that now that the distributions of interest Φ and Θ are both latent variables, the inference process itself will generate all of the distributions we are interested in, so we do not have to run the inference algorithm in a loop to re-estimate the values for Φ as we did before. We may choose to run an EM algorithm to learn the parameters α and β, but this is not explored in this note.

4.3 Evaluation and Implementation Concerns

The inference method proposed for the original model consists of solving an optimization problem for every document to find the variational parameters γ_j and $\pi_{j,t} \forall j \in [1, \ldots, N_j]$, then the results of these M optimization problems to update the model parameters $\phi_i \forall i \in [1, \ldots, K]$. Major criticisms of this method are not just limited to the fact that it does not treat Φ as a random variable—it also contains many calls to the $\Psi(\bullet)$ function which is known to hurt performance. Further, the memory requirements for variational inference are larger than some of the alternative methods, but this method has the advantage of being deterministic (where many alternatives are probabilistic).

Other arguments (such as the ones presented in Teh et al. [11]) criticize the method based on its inaccuracy: the strong independence assumption introduced in the fully factorized variational distribution eliminates the strong dependence between the latent variables $\mathbf{Z}$, Θ, and Φ which can lead to inaccurate estimates of the posterior due to the loose upper bound on the negative log likelihood.

5 Collapsed Gibbs Sampling for LDA

Another general approach for posterior inference with intractable distributions is to appeal to Markov-chain Monte Carlo methods. Gibbs sampling is one such algorithm, but there are many others (such as Metropolis Hastings). Here, the goal is to produce samples from a distribution that is hard to sample from directly. The Gibbs sampling method is applicable when the distribution to sample from is intractable, but the *conditional distribution* of a specific variable *given all the others* can be computed easily. The algorithm proceeds in "rounds" (sometimes called "epochs") where each latent variable is sampled in turn conditioned on the current values of all of the other latent variables. This process "sweeps" through all of the latent variables and the round ends when all of the latent variables have been re-sampled according to their full conditional distributions. This strategy can be shown to eventually (after some amount of burn-in time) result in accurate samples from the true posterior distribution.

In the case of LDA, we can make a further simplification that will enable the sampler to converge faster. Because the priors over Φ and Θ are *conjugate*, they can be *integrated out* of the joint distribution, resulting in a distribution $P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta)$, which we can then use to build a Gibbs sampler over $\mathbf{Z}$. This process of integrating out the conjugate priors is often called "collapsing" the priors, and thus a sampler built in this way is often called a "collapsed Gibbs sampler". In this section, we will show how to derive a collapsed Gibbs sampler for LDA and also introduce a variation of the algorithm that, while approximate, can leverage parallel computation resources (either on a single system with multiple threads or multiple systems).

These methods are generally attractive in the sense that they are typically easily implemented, and the sampling method itself has nice guarantees about convergence to the true posterior. Unfortunately, because it is a randomized algorithm, it does not have the deterministic guarantees that the previous variational inference method does.

5.1 Original Method

The general idea behind Gibbs sampling methods is to take repeated samples from the *full conditional* distribution for each latent variable in turn, iteratively. In this way, the current assignments for all other latent variables influence the sampling for the "replacement" latent variable in the current iteration.

One could in theory apply Gibbs sampling directly to LDA, but this would exhibit slow mixing due to the interactions between the latent variables. Instead, Griffiths and Steyvers [5] derive a sampler that integrates out the variables Θ and Φ by taking advantage of the fact that the Dirichlet is the conjugate prior of the multinomial in order to simplify the integrals. Then, based on the posterior distribution of the latent topic assignments $\mathbf{Z}$, we can get estimates for the distributions Θ and Φ.

The key observation is this: suppose that we have the joint distribution of the corpus and the topic assignments $P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta)$ with the parameters Θ and Φ integrated out. Then, because $\mathbf{W}$ is observed, we simply need to sample each $z_{m,n}$ in turn, given the full conditional consisting of all other topic assignments $\mathbf{Z}_{\neg m,n}$. We can see that

$$\begin{aligned} P(z_{m,n} = k \mid \mathbf{Z}_{\neg m,n}, \mathbf{W}, \alpha, \beta) &= \frac{P(z_{m,n} = k, \mathbf{Z}_{\neg m,n}, \mathbf{W} \mid \alpha, \beta)}{P(\mathbf{Z}_{\neg m,n}, \mathbf{W} \mid \alpha, \beta)} \\ &\propto P(z_{m,n} = k, \mathbf{Z}_{\neg m,n}, \mathbf{W} \mid \alpha, \beta) \end{aligned} \tag{50}$$

Thus, with just a definition of $P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta)$, we can derive a Gibbs sampler that iterates over the topic assignments $z_{j,t}$. One can find this distribution by performing the integration over Θ and Φ, which we will show in the next section.

5.2 Integrating Out the Priors in LDA

We start by noting that

$$P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta) = P(\mathbf{Z} \mid \alpha, \beta) P(\mathbf{W} \mid \mathbf{Z}, \alpha, \beta) \tag{51}$$

by the definition of conditional probability, and

$$= P(\mathbf{Z} \mid \alpha) P(\mathbf{W} \mid \mathbf{Z}, \beta) \tag{52}$$

according to our modeling assumptions. We can now focus on these two terms individually.

Starting with $P(\mathbf{Z} \mid \alpha)$, we have that

$$P(\mathbf{Z} \mid \alpha) = \int P(\Theta \mid \alpha) P(\mathbf{Z} \mid \Theta) d\Theta \tag{53}$$

from our model definition. Focusing on the prior term $P(\Theta \mid \alpha)$, we have

$$P(\Theta \mid \alpha) = \prod_{j=1}^{M} P(\theta_j \mid \alpha) = \prod_{j=1}^{M} \frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_{j,i}^{\alpha_i - 1}. \tag{54}$$

We also know that

$$P(\mathbf{Z} \mid \Theta) = \prod_{j=1}^{M} \prod_{t=1}^{N_j} \theta_{j,z_{j,t}} = \prod_{j=1}^{M} \prod_{i=1}^{K} \theta_{j,i}^{\sigma_{j,i}} \tag{55}$$

where

$$\sigma_{j,k} = \sum_{t=1}^{N_j} \mathbb{1}(z_{j,t} = k) \tag{56}$$

is the number of times we observe topic k in document j. Combining these two distributions, we can see that

$$\int P(\Theta \mid \alpha) P(\mathbf{Z} \mid \Theta) d\Theta = \prod_{j=1}^{M} \int \frac{1}{B(\alpha)} \prod_{i=1}^{K} \theta_{j,i}^{\sigma_{j,i}+\alpha_i-1} d\theta_j \tag{57}$$

$$= \prod_{j=1}^{M} \frac{1}{B(\alpha)} \int \prod_{i=1}^{K} \theta_{j,i}^{\sigma_{j,i}+\alpha_i-1} d\theta_j. \tag{58}$$

We can now apply the property we derived in Equation 8 to show that

$$P(\mathbf{Z} \mid \alpha) = \prod_{j=1}^{M} \frac{B(\alpha + \sigma_j)}{B(\alpha)}, \tag{59}$$

where σ_j is the count vector over topics in document j.

Turning our attention to $P(\mathbf{W} \mid \mathbf{Z}, \beta)$, we see a similar story. Namely

$$P(\mathbf{W} \mid \mathbf{Z}, \beta) = \int P(\Phi \mid \beta) P(\mathbf{W} \mid \mathbf{Z}, \Phi) d\Phi. \tag{60}$$

We also know that

$$P(\Phi \mid \beta) = \prod_{i=1}^{K} P(\phi_i \mid \beta) = \prod_{i=1}^{K} \frac{1}{B(\beta)} \prod_{r=1}^{V} \phi_{i,r}^{\beta_r - 1}, \tag{61}$$

and

$$P(\mathbf{W} \mid \mathbf{Z}, \Phi) = \prod_{j=1}^{M} \prod_{t=1}^{N_j} P(w_{j,t} \mid \phi_{z_{j,t}}) = \prod_{i=1}^{K} \prod_{r=1}^{V} \phi_{i,r}^{\delta_{i,r}}, \tag{62}$$

where $\delta_{i,r}$ is defined analogously to $\sigma_{j,k}$ as the number of times word type r appears in topic i. Thus,

$$\int P(\phi \mid \beta) P(\mathbf{W} \mid \mathbf{Z}, \Phi) d\Phi = \prod_{i=1}^{K} \int \frac{1}{B(\beta)} \prod_{r=1}^{V} \phi_{i,r}^{\delta_{i,r}+\beta_r-1} d\phi_i \tag{63}$$

$$= \prod_{i=1}^{K} \frac{1}{B(\beta)} \int \prod_{r=1}^{V} \phi_{i,r}^{\delta_{i,r}+\beta_r-1} d\phi_i. \tag{64}$$

We can again apply the property we derived in Equation 8 to show that

$$P(\mathbf{W} \mid \mathbf{Z}, \beta) = \prod_{i=1}^{K} \frac{B(\beta + \delta_i)}{B(\beta)} \tag{65}$$

where δ_i is the count vector associated with topic i. This leads to a *collapsed* joint probability definition of

$$P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta) = \left(\prod_{j=1}^{M} \frac{B(\alpha + \sigma_j)}{B(\alpha)}\right)\left(\prod_{i=1}^{K} \frac{B(\beta + \delta_i)}{B(\beta)}\right). \tag{66}$$

5.3 Deriving the Collapsed Gibbs Sampler

We showed in the previous section that we can integrate out the priors over Φ and Θ and arrive at

$$\begin{aligned} P(\mathbf{W}, \mathbf{Z} \mid \alpha, \beta) &= \int_{\Theta}\int_{\Phi} \prod_{i=1}^{K} P(\phi_i \mid \beta) \prod_{j=1}^{M} P(\theta_j \mid \alpha) \prod_{t=1}^{N_j} P(z_{j,t} \mid \theta_j) P(w_{j,t} \mid \phi_{z_{j,t}}) d\Theta d\Phi \\ &= \left(\prod_{j=1}^{M} \frac{B(\alpha + \sigma_j)}{B(\alpha)}\right)\left(\prod_{i=1}^{K} \frac{B(\beta + \delta_i)}{B(\beta)}\right) \qquad (67) \\ &= \left(\frac{\Gamma\left(\sum_{i=1}^{K} \alpha_i\right)}{\prod_{i=1}^{K} \Gamma(\alpha_i)}\right)^{M} \prod_{j=1}^{M} \frac{\prod_{i=1}^{K} \Gamma(\sigma_{j,i} + \alpha_i)}{\Gamma\left(\sum_{i=1}^{K} \sigma_{j,i} + \alpha_i\right)} \\ &\quad \times \left(\frac{\Gamma\left(\sum_{r=1}^{V} \beta_r\right)}{\prod_{r=1}^{V} \Gamma(\beta_r)}\right)^{K} \prod_{i=1}^{K} \frac{\prod_{r=1}^{V} \Gamma(\delta_{i,r} + \beta_r)}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r} + \beta_r\right)} \qquad (68) \end{aligned}$$

where $\sigma_{j,i}$ is the number of times topic i has been chosen for words in document j, and $\delta_{i,r}$ is the number of times topic i has been chosen for word type r. We can now derive the sampler on the basis of Equation 50.

We start by dropping the two coefficients in front of the product over the documents and the product over the topics by noting that these do not depend on the specific topic assignment $z_{m,n}$. We then have that

$$P(\mathbf{W}, z_{m,n} = k, \mathbf{Z}_{\neg m,n} \mid \alpha, \beta) \propto \prod_{j=1}^{M} \frac{\prod_{i=1}^{K} \Gamma(\sigma_{j,i} + \alpha_i)}{\Gamma\left(\sum_{i=1}^{K} \sigma_{j,i} + \alpha_i\right)} \times \prod_{i=1}^{K} \frac{\prod_{r=1}^{V} \Gamma(\delta_{i,r} + \beta_r)}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r} + \beta_r\right)}. \tag{69}$$

Now we can extract only those terms that depend on the the position (m, n) we are sampling for (as all others are constants as we vary $z_{m,n}$):

$$\propto \frac{\prod_{i=1}^{K} \Gamma(\sigma_{m,i} + \alpha_i)}{\Gamma\left(\sum_{i=1}^{K} \sigma_{m,i} + \alpha_i\right)} \times \prod_{i=1}^{K} \frac{\Gamma(\delta_{i,w_{m,n}} + \beta_{w_{m,n}})}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r} + \beta_r\right)} \tag{70}$$

Now let's let

$$\sigma_{j,i}^{\neg m,n} = \begin{cases} \sigma_{j,i} - 1 & \text{if } j = m \text{ and } i = z_{m,n} \\ \sigma_{j,i} & \text{otherwise} \end{cases} \tag{71}$$

be the number of times we see topic i in document j, but excluding the current assignment $z_{m,n}$. Similarly, let

$$\delta_{i,r}^{\neg m,n} = \begin{cases} \delta_{i,r} - 1 & \text{if } i = z_{m,n} \text{ and } r = w_{m,n} \\ \delta_{i,r} & \text{otherwise.} \end{cases} \tag{72}$$

We can then rewrite the right hand side of Equation 70 above as

$$\frac{\left(\prod_{i=1;i\neq k}^{K} \Gamma(\sigma_{m,i}^{\neg m,n} + \alpha_i)\right) \Gamma(1 + \sigma_{m,k}^{\neg m,n} + \alpha_k)}{\Gamma\left(1 + \sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i\right)} \times \prod_{i=1;i\neq k}^{K} \frac{\Gamma(\delta_{i,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r}^{\neg m,n} + \beta_r\right)} \times \frac{\Gamma(1 + \delta_{k,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})}{\Gamma\left(1 + \sum_{r=1}^{V} \delta_{k,r}^{\neg m,n} + \beta_r\right)}. \tag{73}$$

Now, we use the property of the gamma function that

$$\Gamma(1 + x) = x\Gamma(x) \tag{74}$$

to rewrite the terms in Equation 73, which then becomes

$$\frac{\left(\prod_{i=1;i\neq k}^{K} \Gamma(\sigma_{m,i}^{\neg m,n} + \alpha_i)\right) \Gamma(\sigma_{m,k}^{\neg m,n} + \alpha_k)(\sigma_{m,k}^{\neg m,n} + \alpha_k)}{\Gamma\left(\sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i\right)\left(\sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i\right)} \times \prod_{i=1;i\neq k}^{K} \frac{\Gamma(\delta_{i,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r}^{\neg m,n} + \beta_r\right)} \times \frac{\Gamma(\delta_{k,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})(\delta_{k,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})}{\Gamma\left(\sum_{r=1}^{V} \delta_{k,r}^{\neg m,n} + \beta_r\right)\left(\sum_{r=1}^{V} \delta_{k,r}^{\neg m,n} + \beta_r\right)}. \tag{75}$$

We can now move the gamma functions back inside the products to make them occur across all topics again, resulting in

$$\frac{\prod_{i=1}^{K} \Gamma(\sigma_{m,i}^{\neg m,n} + \alpha_i)}{\Gamma\left(\sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i\right)} \times \frac{\sigma_{m,k}^{\neg m,n} + \alpha_k}{\sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i} \times \prod_{i=1}^{K} \frac{\Gamma(\delta_{i,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}})}{\Gamma\left(\sum_{r=1}^{V} \delta_{i,r}^{\neg m,n} + \beta_r\right)} \times \frac{\delta_{k,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}}}{\sum_{r=1}^{V} \delta_{k,r}^{\neg m,n} + \beta_r}. \tag{76}$$

Finally, we can note that we can drop terms 1 and 3 in this expression as they do not depend on the assignment of $z_{m,n} = k$. Thus, we arrive at the sampling probability

$$P(z_{m,n} = k \mid \mathbf{Z}_{\neg m,n}, \mathbf{W}, \alpha, \beta) \propto \frac{\sigma_{m,k}^{\neg m,n} + \alpha_k}{\sum_{i=1}^{K} \sigma_{m,i}^{\neg m,n} + \alpha_i} \times \frac{\delta_{k,w_{m,n}}^{\neg m,n} + \beta_{w_{m,n}}}{\sum_{r=1}^{V} \delta_{k,r}^{\neg m,n} + \beta_r}. \tag{77}$$

This can be efficiently implemented by caching the (sparse) matrices σ and δ. To resample each $z_{m,n}$, simply subtract one from the appropriate entries in σ and δ and sample according to Equation 77, adding one to the counts that correspond to the entries associated with the new assignment of $z_{m,n}$.

To recover the distributions Φ and Θ, we can simply utilize a point estimate based on the current state of **Z** in the Markov chain. Namely,

$$\theta_{j,k} \approx \frac{\sigma_{j,k} + \alpha_k}{\sum_{i=1}^{K} \sigma_{j,i} + \alpha_i} \tag{78}$$

$$\phi_{k,v} \approx \frac{\delta_{k,v} + \beta_v}{\sum_{r=1}^{V} \delta_{k,r} + \beta_r}. \tag{79}$$

5.3.1 Evaluation and Implementation Concerns

By simply maintaining counts $\sigma_{j,i}$ and $\delta_{i,r}$, we can easily write the process that samples each topic assignment in turn. In general, these collapsed Gibbs sampling methods use the least amount of memory in comparison to the other inference methods and are incredibly easy to implement, leading to widespread adoption.

The main issues posed for the collasped Gibbs sampling methods are the convergence rate (and evaluating the convergence of the Markov chain) and noisiness in the samples. Evaluating the convergence of the chain is typically done by tracking $\log P(\mathbf{W} \mid \mathbf{Z})$ at each iteration of the sampler (as in, for a fixed value of **Z** at the end of an "epoch" of sampling). Once this quantity converges, the chain is considered to be have converged.

However, even with this convergence evaluation, one needs to take care to allow the chain to run for a certain number of iterations to avoid taking a sample that is heavily auto-correlated. In practice, most papers use a "burn-in" period of a fixed number of iterations before taking any samples from the chain. Even then, care must still be exercised when taking these samples: while it may be tempting to take just a single sample, it is possible for this sample to contain noise. To remedy this, one may take several samples from the Markov chain and average them to arrive at the final estimates for Θ and Φ. Again, when taking these samples, care must be exercised to avoid auto-correlation (typically another, but shorter, "burn-in" period is undertaken between consecutive samples).

5.4 Distributed/Parallel Method

Inference in LDA is often an expensive process. Indeed, in most of the literature the algorithm is only ever run on small corpora, or with an extremely limited vocabulary. However, if one wanted to utilize LDA as part of a search engine, for instance, more efficient methods are necessary without placing these harsh restrictions on the data considered by the model. Following this line of thinking, Newman et al. [10] derive algorithms for "approximate distributed" versions of the LDA model and the hierarchical Dirichlet process. We focus on their model for LDA, called AD-LDA, here (we ignore HD-LDA as it presents a modified version of the graphic model and their methods for HDP as they do not pertain to LDA directly).

Parallelization over the non-collapsed (or partially collapsed) Gibbs sampling can be performed relatively easily due to the independence between variable samplings. However, the non-collapsed Gibbs sampling method is very rarely used for LDA as it exhibits very slow convergence compared to the collapsed version introduced by Griffiths and Steyvers [5]. Therefore, it's desirable to attempt to parallelize the collapsed sampler—but this is difficult because the sampler is inherently sequential, with each sample depending on the outcome of the previous sample.

To address this problem, the authors propose to relax this assumption, reasoning that if two processors are sampling topics in different documents and different words that the influence of one outcome on the other ought to be small. Their algorithm first partitions the data (by document) among the processors to be used. Each processor is given local counts $\sigma_{j,i}^{(p)}$ and $\delta_{i,r}^{(p)}$. Note that, because of the partitioning by document, $\sigma_{j,i}^{(p)} = \sigma_{j,i}$ and will be consistent globally with the assignments $\mathbf{Z}$. The problem comes with $\delta_{i,r}^{(p)}$, which will be inconsistent with the global assignments $\mathbf{Z}$ and cannot be updated across processors safely without introducing race conditions. The AD-LDA algorithm thus has a "reduction" step, where the processor-local counts $\delta_{i,r}^{(p)}$ are merged together to be consistent with the global topic assignments $\mathbf{Z}$. The pseudocode for this algorithm is given as Algorithm 1.

Algorithm 1 AD-LDA

```
repeat
    for each processor $p$ in parallel do
      Copy global counts: $\delta_{i,r}^{(p)} \leftarrow \delta_{i,r}$
      Sample $\mathbf{Z}_p$ locally using CGS on $p$'s partition using $\delta_{i,r}^{(p)}$ and $\sigma_{j,i}$.
    end for
      Wait for all $p$ to finish.
      Update global counts: $\delta_{i,r} \leftarrow \delta_{i,r} + \sum_p (\delta_{i,r}^{(p)} - \delta_{i,r})$
until convergence/termination criteria satisfied
```

5.4.1 Evaluation and Implementation Concerns

Because of the relaxation of the sequential nature of the sampler, this algorithm is no longer taking samples from the true posterior asymptotically, but from something that approximates it. A natural question, then, is to what posterior does this method approach, and is it reasonably close to the actual posterior desired? The authors address this question by appealing to two major evaluation methods: perplexity of a test set after training both normal LDA and AD-LDA on the same training set, and an evaluation of a ranking problem on the 20 newsgroups dataset.

We first address their results on perplexity. They establish that the perplexity of their model does not deviate much (if at all) from the perplexity achieved by a regular LDA model. Further, it does not deviate much between the number of processors p used (in their experiments, 10 and 100). Perplexity is a poor measure for the performance of topic models in general [3], but it may be reasonable to assume that similar perplexity means similar models were learned.

Their results on a ranking problem are perhaps more interesting. Their setup was to train both LDA and AD-LDA on a training corpus from the 20 newsgroups dataset, and then test against a test set. Each test document is treated as a query, and the goal is to retrieve relevant documents from the training set with respect to this query. Ranking was performed by evaluating the probability of the query document given the training document's topic proportions θ_j and the topics Φ found by the model. Here, the difference between their model and the traditional LDA model were more apparent, with their model generally lagging behind the LDA model in mean average precision (MAP) and mean area under ROC curve until about 30–40 iterations. This to me is a sign that the AD-LDA inference deviates significantly from the LDA model (at least in the first few iterations). There is an open question as to how exactly their assumptions that the non-sequential sampling would have minimal impact were violated by the real data.

6 Collapsed Variational Inference for LDA

Just as we can "collapse" the priors before deriving a Gibbs sampler for LDA, we can also "collapse" the priors before deriving a variational inference algorithm. In this section, we will introduce the idea of "collapsed variational inference" for LDA, which leverages the conjugacy of the Dirichlet priors to integrate them out of the joint distribution. Then, a mean-field variational distribution can be introduced to approximate the distribution $P(\mathbf{Z} \mid \mathbf{W}, \alpha, \beta)$. This approach to variational inference can be shown to produce a better approximation to the posterior distribution due to the strictly weaker independence assumptions introduced in the variational distribution and results in an algorithm that maintains the deterministic nature of the original variational inference algorithm, but with significantly convergence time over traditional variational inference. In practice, these methods can derive deterministic inference algorithms that are more accurate and faster than even collapsed Gibbs sampling.

6.1 Original Method

Teh, Newman, and Welling [11] formulate their algorithm, called Collapsed Variational Bayes (CVB), as an attempt to address the problem of inaccuracy in the original variational inference algorithms presented by Blei et al. [2]. Their algorithm purports to combine the advantages of the determinism of the variational inference methods with the accuracy of the collapsed Gibbs sampling methods.

The main idea they present is that one can marginalize out Θ and Φ before creating a variational approximation over $\mathbf{Z}$. Thus, their variational distribution looks like

$$\hat{q}(\mathbf{Z}) = \prod_{j=1}^{M} \prod_{t=1}^{N_j} \hat{q}_j(z_{j,t} \mid \hat{\gamma}_{j,t}) \tag{80}$$

where $\hat{\gamma}_{j,t}$ are the parameters of the variational multinomial $\hat{q}_j(\bullet)$ over the topic assignments for the t-th word in the j-th document. The graphical model for this variational distribution is depicted in Figure 5. Because this makes a strictly weaker independence assumption than that of the original variational inference algorithm, they establish their method gives a better approximation to the posterior as well.

Minimizing the KL-divergence between their variational distribution $q_j(z)$ and the true (collapsed) posterior $p_j(z)$, they arrive at the following updating equation for $\hat{q}(z_{m,n} = k \mid \hat{\gamma}_{m,n}) = \hat{\gamma}_{m,n,k}$ as (letting

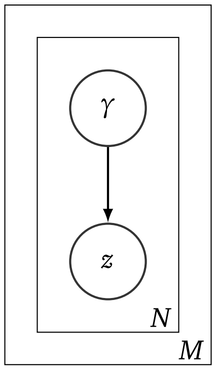

Figure 5: Graphical model for the factorized variational distribution for collapsed variational bayes [11].

$w_{m,n} = v$):

$$\hat{\gamma}_{m,n,k} = \frac{\exp\left(E_{\hat{q}(\mathbf{Z}_{\neg m,n})}\left[P(\mathbf{W}, \mathbf{Z}_{\neg m,n}, z_{m,n} = k \mid \alpha, \beta)\right]\right)}{\sum_{i=1}^{K} \exp\left(E_{\hat{q}(\mathbf{Z}_{\neg m,n})}\left[P(\mathbf{W}, \mathbf{Z}_{\neg m,n}, z_{m,n} = i \mid \alpha, \beta)\right]\right)} \tag{81}$$

$$= \frac{\exp\left(E_{\hat{q}(\mathbf{Z}_{\neg m,n})}\left[\log(\alpha_k + \sigma_{m,k}^{\neg m,n}) + \log(\beta_v + \delta_{k,v}^{\neg m,n}) - \log(\sum_{r=1}^{V} \beta_r + \delta_{k,r}^{\neg m,n})\right]\right)}{\sum_{i=1}^{K} \exp\left(E_{\hat{q}(\mathbf{Z}_{\neg m,n})}\left[\log(\alpha_i + \sigma_{m,i}^{\neg m,n}) + \log(\beta_v + \delta_{i,v}^{i,\neg m,n}) - \log(\sum_{r=1}^{V} \beta_r + \delta_{i,r}^{\neg m,n})\right]\right)}. \tag{82}$$

The pain point is in calculating the expectations in the above formula. While the authors give an exact evaluation, this exact evaluation is computationally infeasible and thus they appeal to a "Gaussian approximation" for the expectations.

They begin by noticing that $\sigma_{m,k}^{\neg m,n} = \sum_{t \neq n} \mathbb{1}(z_{m,t} = k)$ is a sum of Bernoulli random variables with a mean parameter $\hat{\gamma}_{m,t,k}$ and thus can be accurately approximated with a Gaussian with mean $E_{\hat{q}}[\sigma_{m,k}^{\neg m,n}] = \sum_{t \neq n} \hat{\gamma}_{m,t,k}$ and variance $Var_{\hat{q}}[\sigma_{m,k}^{\neg m,n}] = \sum_{t \neq n} \hat{\gamma}_{m,t,k}(1 - \hat{\gamma}_{m,t,k})$. They approximate the logarithms with a second order Taylor series and evaluate the expectation under the Gaussian approximation (they argue that their second order approximation is "good enough" because $E_{\hat{q}}[\sigma_{m,k}^{\neg m,n}] \gg 0$). Using this approximation, the updates for CVB become:

$$\hat{\gamma}_{m,n,k} \propto \left(E_{\hat{q}}[\sigma_{m,k}^{\neg m,n}] + \alpha_k\right) \frac{\left(E_{\hat{q}}[\delta_{k,v}^{\neg m,n}] + \beta_v\right)}{\left(\sum_{r=1}^{V} E_{\hat{q}}[\delta_{k,r}^{\neg m,n}] + \beta_r\right)}$$
$$\times \exp\left(-\frac{Var_{\hat{q}}(\sigma_{m,k}^{\neg m,n})}{2\left(\alpha_k + E_{\hat{q}}[\sigma_{m,k}^{\neg m,n}]\right)^2} - \frac{Var_{\hat{q}}(\delta_{k,v}^{\neg m,n})}{2\left(\beta_v + E_{\hat{q}}[\delta_{k,v}^{\neg m,n}]\right)^2} + \frac{Var_{\hat{q}}\left(\sum_{r=1}^{V} \delta_{k,r}^{\neg m,n}\right)}{2\left(\sum_{r=1}^{V} \beta_r + \delta_{k,r}^{\neg m,n}\right)^2}\right) \tag{83}$$

The similarity with CGS as given in Equation 77 is apparent. In particular, the first term is just CGS with the counts σ and δ replaced with their expectations with respect to the variational distribution $\hat{q}(\bullet)$, and the second term can be interpreted as "correction factors" accounting for the variance [11].

The distributions of interest, Θ and Φ, can be extracted using a similar method to that used in CGS, but by using the expectations of the counts with respect to $\hat{q}(\bullet)$ instead of the counts directly. Namely,

$$\theta_{j,k} \approx \frac{(E_{\hat{q}}[\sigma_{j,k}] + \alpha_k)}{\left(\sum_{i=1}^{K} E_{\hat{q}}[\sigma_{j,i}] + \alpha_i\right)} \tag{84}$$

$$\phi_{k,v} \approx \frac{(E_{\hat{q}}[\delta_{k,v}] + \beta_v)}{\left(\sum_{r=1}^{V} E_{\hat{q}}[\delta_{k,r}] + \beta_r\right)}. \tag{85}$$

6.1.1 Evaluation and Implementation Concerns

While the authors proved that CVB has a better posterior approximation as compared to the variational Bayes (VB) inference of Blei et al. [2], they also provided some empirical results to justify their method.

To evaluate just how much better the CVB approximation was when compared to VB, they looked at the variational bounds for VB vs CVB at different numbers of iterations. As expected, the variational bound for CVB was strictly better than VB across all iterations for both datasets they investigated. They also ran both algorithms to convergence several times and looked at the distributions of the variational bounds generated by both algorithms, finding that CVB consistently outperformed VB as to be expected.

To see how close to the real posterior their CVB approximation got, they compared CVB to CGS in terms of test set word log probabilities for different numbers of iterations. The findings were mixed: on one dataset their method converged in approximately 30 iterations, but CGS surpassed it at about 25 iterations and continued to get better results as far as 100 iterations. In the other dataset, their algorithm converged in approximately 20 iterations and was only surpassed by CGS at approximately 40 iterations. In general, it appears that CGS will converge to a strictly better approximation of the posterior, but this is expected as it asymptotically should approach the real posterior. CVB has the potential to converge faster than either CGS and VB.

CVB may be implemented by keeping track of the mean and variance of the counts σ and δ with respect to $\hat{q}(\bullet)$, and lacks the repeated calls to the $\Psi(\bullet)$ function that plague standard VB. Furthermore, they can store only one copy of the variational parameter for every unique document-word pair (j, r) and have the same memory requirements as VB. The authors claim that CVB performs less operations per iteration than CGS, but that the constants for CGS are smaller. However, as we will see in the next section, the operations per iteration can be dropped even further by using a simpler approximation.

6.2 Lower-order Approximations

Asuncion et al. [1] propose an alternative method for setting $\hat{\gamma}_{m,n,k}$ that uses only the zeroth-order information instead of the second-order information used in Equation 83. Namely,

$$\hat{\gamma}_{m,n,k} \propto \left(E_{\hat{q}}[\sigma_{m,k}^{\neg m,n}] + \alpha_k\right) \frac{\left(E_{\hat{q}}[\delta_{k,v}^{\neg m,n}] + \beta_v\right)}{\left(\sum_{r=1}^{V} E_{\hat{q}}[\delta_{k,r}^{\neg m,n}] + \beta_r\right)}. \tag{86}$$

They refer to this algorithm as CVB0, and note its extreme similarity to CGS's sampling in Equation 77. The only difference between these two algorithms is that CGS maintains a topic assignment

$z_{m,n}$ for each token in each document, while CVB0 maintains a distribution over topic assignments for each token (m, n).

6.2.1 Evaluation and Implementation Concerns

CVB0 can be implemented in a very similar way to CVB, except that statistics for the variance terms need not be calculated. The authors propose that their algorithm is more performant than CVB for this reason—lack of variance statistics and calls to $\exp(\bullet)$ should make the iterations faster.

They evaluate their results in this respect by comparing how long it takes, in seconds, for each algorithm to reach a certain perplexity level on the training data. The table is replicated here as Table 1. The find that, across all three of their datasets, CVB0 converges faster or as fast as CGS. This makes intuitive sense: CVB0 is essentially CGS but with propagation of uncertainty throughout the sampling process. The authors also provide results for a parallelized version of CVB0, which (while little detail was given) is very likely to be similar to the work for AD-LDA in Newman et al. [10]. We call it "AD-CVB0" here.

	MED	**KOS**	**NIPS**
VB	151.6	73.8	126.0
CVB	25.1	9.0	21.7
CGS	18.2	3.8	12.0
CVB0	9.5	4.0	8.4
AD-CVB0	2.4	1.5	3.0

Table 1: Timing results, in seconds, for the different inference algorithms discussed so far to reach a fixed perplexity threshold set in Asuncion et al. [1]

7 An Aside: Analogy to k-means vs EM Clustering

Consider for a moment the task of clustering tokens into K distinct clusters. Two general approaches exist: one is to suggest that each individual token belongs to a single cluster and perform a "hard assignment" of that token to a cluster. This has an advantage that it's easy to understand, but a disadvantage in that it loses out on the uncertainty that comes with that cluster assignment. Instead, we could choose to do a "soft assignment" of the token to the clusters, where the cluster assignment for each token n can be viewed as a vector of probabilities v_n that, when indexed by a cluster id k, yields $v_{n,k} = P(z_n = k)$, the probability of membership to that cluster.

CGS performs the first option, where each draw from Equation 77 assigns a distinct topic to the n-th token in document m, where as CVB (and, by extension, CVB0) performs the second option where "cluster membership" is "topic membership" and is indexed by $\hat{\gamma}_{m,n,k}$. Thus, it is interesting to think about CGS as a "hard assignment" token clustering algorithm and CVB(0) as a "soft assignment" clustering algorithm.

8 Online (Stochastic) Methods for Inference in LDA

Every method discussed thus far has been a *batch learning* method: take each of the documents in D and update your model parameters by cycling through every document $d_j \in D$. It is well established in the machine learning community that *stochastic* optimization algorithms often outperform batch optimization algorithms. For example, SGD has been shown to significantly outperform L-BFGS for the task of learning conditional random fields[3].

Because the sizes of our text corpora are always growing, it would be incredibly beneficial for us to be able to learn our topic models in an online fashion: treat each document as it is arriving as sampled uniformly at random from the set of all possible documents, and perform inference using just this one example (or a small set of examples in mini-batch setup). In this section, we will explore two new stochastic algorithms for inference in LDA models: stochastic variational inference [6] and stochastic collapsed variational inference [4].

8.1 Stochastic Variational Inference

Stochastic variational inference [6] for LDA is a modification of the variational Bayes inference algorithm presented in Blei et al. [2] to an online setting. They present it in a general form as follows: first, separate the variables of the variational distribution into a set of "local" and "global" variables. The "local" variables should be specific to a data point, and the "global" variables are in some form an aggregation over all of the data points. The algorithm then proceeds as given in Algorithm 2.

Algorithm 2 Stochastic Variational Inference

```
Initialize the global parameters randomly,
Set a step schedule $\rho_t$ appropriately,
repeat
    Sample data point $x_i$ from the data set uniformly,
    Compute local parameters,
    Compute intermediate global parameters as if $x_i$ was replicated $N$ times,
    Update the current estimate of global parameters according to the step schedule $\rho_t$.
until termination criterion met
```

(As a refresher, refer to Figure 4 for the variational parameter definitions.) They begin by separating the variational distribution parameters into the "global" and "local" parameters. For LDA, the global parameters are the topics λ_i, and the local parameters are document-specific topic proportions γ_j and the token-specific topic assignment distribution parameters $\pi_{j,t}$.

Let $\lambda^{(t)}$ be the topics found after iteration t. Then, the algorithm proceeds as follows: sample a document j from the corpus, compute its local variational parameters γ_j and π_j by iterating between the two just as is done in standard variational inference.

[3] http://leon.bottou.org/projects/sgd

Once these parameters are found, they are used to find the "intermediate" global parameters:

$$\hat{\lambda}_{i,r} = \beta_r + M \sum_{t=1}^{N_j} \pi_{j,t,i} \mathbb{1}(w_{j,t} = r) \tag{87}$$

and then we set the global parameters to be an average of the intermediate $\hat{\lambda}_i$ and the global $\lambda_i^{(t)}$ from the previous iteration:

$$\lambda_{i,r}^{(t+1)} = (1 - \rho_t)\lambda_{i,r}^{(t)} + \rho_t \hat{\lambda}_{i,r} \tag{88}$$

This makes intuitive sense and has a strong resemblance to stochastic gradient descent. Thus, the usual considerations for setting a learning rate schedule ρ_t apply. Namely, a schedule that monotonically decreases towards (but never reaching) zero is preferred. In the paper, they suggest a learning rate parameterized by κ and τ as follows:

$$\rho_t = \frac{1}{(t + \tau)^\kappa} \tag{89}$$

though other, equally valid, schedules exist in the literature for stochastic gradient descent. The algorithm, much like SGD, can also be modified to use mini-batches instead of sampling a single document at a time for the global λ update.

8.1.1 Evaluation and Implementation Concerns

The authors discuss two main implementation concerns: setting the batch size and learning rate parameters. They run various experiments, measuring the log probability of at test set, to see the effect of the different choices for these parameters. As a general rule, they find that *large learning rates* ($\kappa \approx 0.9$) and *large batch sizes* are preferred for this method.

Large batch sizes being preferred is an interesting result, which seems to indicate that the gradient given by sampling a single document is much too noisy to actually take a step in the right direction. One interesting question, then, would be to explore the difference this inference method has over just running the original VB algorithm on large batches of documents and merging the models together following a learning rate schedule like ρ_t. This would seem to me to be a very naïve approach, but if it is effective then it would appear to be a much more general result: stochastic *anything* in terms of inference for graphic models could be implemented using this very simple framework.

8.2 Stochastic Collapsed Variational Inference

A natural extension of stochastic variational inference is given by Foulds et al. [4] that attempts to collapse the variational distribution just like CVB from Teh et al. [11]. They additionally use the zeroth-order approximation given in Asuncion et al. [1] for efficiency. An additional interesting feature of this model is that, because of the streaming nature of the algorithm, storing $\gamma_{m,n,k}$ is impossible—working around this limitation can dramatically lower the memory requirements for CVB0.

First, to handle the constraint of not being able to store the $\gamma_{m,n,k}$, the authors only compute it locally for each token that the algorithm examines. Because $\gamma_{m,n,k}$ cannot be stored, they cannot remove the counts for assignment (m,n) from the Equation 86. Thus, they determine $\gamma_{m,n,k}$ with the following approximation (denoting $w_{m,n}$ as v):

$$\gamma_{m,n,k} \propto \left(E_{\hat{q}}[\sigma_{m,k}] + \alpha_k\right) \frac{\left(E_{\hat{q}}[\delta_{k,v}] + \beta_v\right)}{\left(\sum_{r=1}^{V} E_{\hat{q}}[\delta_{k,r}] + \beta_r\right)} \tag{90}$$

Once $\gamma_{m,n,k}$ is computed for a given token, it can be used to update the statistics $E_{\hat{q}}[\sigma_{m,k}]$ and $E_{\hat{q}}[\delta_{k,v}]$ (though the latter is done only after a minibatch is completed using aggregated statistics across the whole minibatch). In particular, the updating equations are

$$E_{\hat{q}}[\sigma_m^k] = (1 - \rho_t^{(\sigma)}) E_{\hat{q}}[\sigma_{j,k}] + \rho_t^{(\sigma)} N_j \gamma_{j,m,n} \tag{91}$$

$$E_{\hat{q}}[\delta_r^k] = (1 - \rho_t^{(\delta)}) E_{\hat{q}}[\delta_{k,r}] + \rho_t^{(\delta)} \hat{\delta}_{k,r} \tag{92}$$

where $\rho^{(\sigma)}$ and $\rho^{(\delta)}$ are separate learning schedules for σ and δ, respectively, and $\hat{\delta}$ is the aggregated statistics for δ with respect to the current minibatch. The algorithm pseudocode is given as Algorithm 3.

Algorithm 3 Stochastic Collapsed Variational Bayes

```
Randomly initialize $\delta_{i,r}$ and $\sigma_{j,i}$.
**for** each minibatch $D_b$ **do**
    $\hat{\delta}_r^i = 0$
    **for** document $j$ in $D_b$ **do**
        **for** $b \geq 0$ "burn-in" phases **do**
            **for** $t \in [1, \ldots, N_j]$ **do**
                Update $\gamma_{j,t,i}$ via Equation 90
                Update $\sigma_{j,i}$ via Equation 91
            **end for**
        **end for**
        **for** $t \in [1, \ldots, N_j]$ **do**
            Update $\gamma_{j,t,i}$ via Equation 90
            Update $\sigma_{j,i}$ via Equation 91
            $\hat{\delta}_{i,w_{j,t}} \leftarrow \hat{\delta}_{i,w_{j,t}} + N\gamma_{j,t,i}$ where $N$ is the total number of tokens in the corpus
        **end for**
        Update $\delta_{i,r}$ via Equation 92
    **end for**
**end for**
```

8.2.1 Evaluation and Implementation Concerns

To evaluate their algorithm, the authors employed three datasets and again measured the test-set log likelihood at different intervals. In this paper, they chose *seconds* instead of iterations, which is a better choice when comparing methods based on their speed. They compared against SVB with appropriate priors set for both algorithms (using the shift suggested by Asuncion et al. [1]). On one dataset (PubMed), they found that the two algorithms behaved mostly similar after about one hour, with SCVB0 outperforming before that time. In the other two datasets (New York Times and Wikipedia), SCVB0 outperformed SVB.

They also performed a user study to evaluate their method in the vein of the suggestions of Chang et al. [3] and found that their algorithm found better topics with significance level $\alpha = 0.05$ on a test where the algorithms were run for 5 seconds each, and with the same significance level on another study where the algorithms were run for a minute each.

9 But Does It Matter? (Conclusion)

Perhaps the most important question that this survey raises is this: does your choice of inference method matter? In Asuncion et al. [1], a systematic look at each of the learning algorithms presented in this survey is undertaken, with interesting results that suggest a shift of the hyperparameters in the uncollapsed VB algorithms by -0.5. This insight prompted a more careful look at the importance of the parameters on the priors in the LDA model for the different inference methods.

Their findings essentially prove that there is little difference in the quality of the output of the different inference algorithms if one is careful with the hyperparameters. First, if using uncollapsed VB, it is important to apply the shift of -0.5 to the hyperparameters. Across the board, it is also very important to set the hyperparameters correctly, and they find that simply using Minka's fixed point iterations [9] is not necessarily guaranteed to find you the most desireable hyperparameter. Instead, the takeaway point is to perform grid search to find suitable settings for the hyperparameters (but this may not always be feasible in practice, so Minka's fixed point iterations are also reasonable).

When doing both the hyperparameter shifting and hyperparameter learning (either via Minka's fixed point iterations or grid search), the difference between the inference algorithms almost vanishes. They verified this across several different datasets (seven in all). Other studies of the influence of hyperparametrs on LDA have been undertaken, such as the Wallach, Mimno, and McCallum [12] study that have produced interesting results on the importance of the priors on the topic model (namely, that an asymmetric prior on Θ and a symmetric prior on ϕ seems to perform the best). It would appear that paying more attention to the parameters of the model, as opposed the mechanism used to learn it, has the largest impact on the overall results of the model.

The main takeaway, then, is that given a choice between a set of different feasible algorithms for approximate posterior inference, *the only preference one should have is toward the method that takes the least amount of time* to achieve a reasonable level of performance. In that regard, the methods based on collapsed variational inference seem to be the most promising, with CVB0 and SCVB0 providing a batch and stochastic version of a deterministic and easy to implement algorithm for approximate posterior inference in LDA.

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